Calculus for AP Physics C Part 2 - Integrals

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Calculus

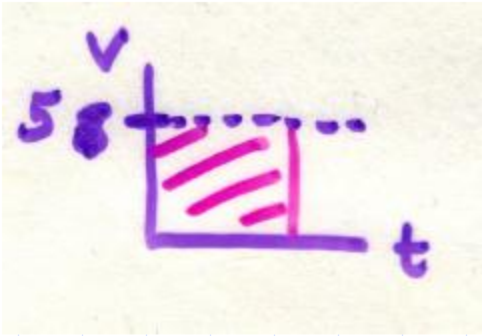
- Differential Calculus = Derivatives (slope)
- Integral Calculus = Integrals (area)

Review:

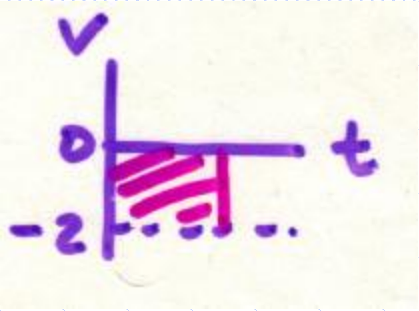
- Inverse operations:
- Add/subtract; multiply/divide sin, inverse sin etc.
- Differentiation and Integration are inverses of each other – they “undo” each other.

In Physics

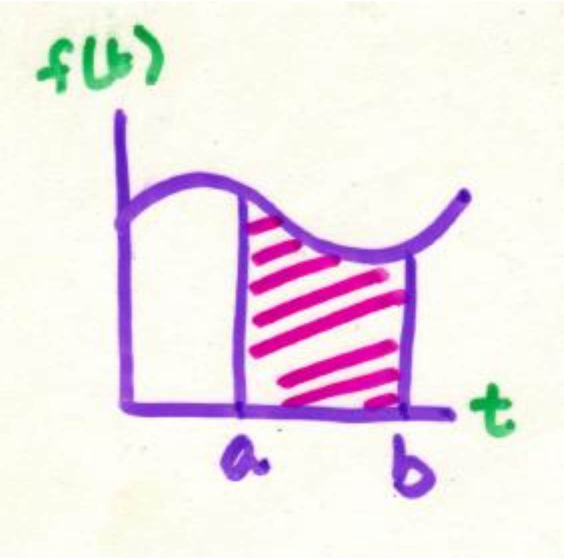
- Velocity-time graph
- Slope (derivative) = acceleration
- Area under the graph (integral) = Δx (displacement)
  - $\Delta x = vt$

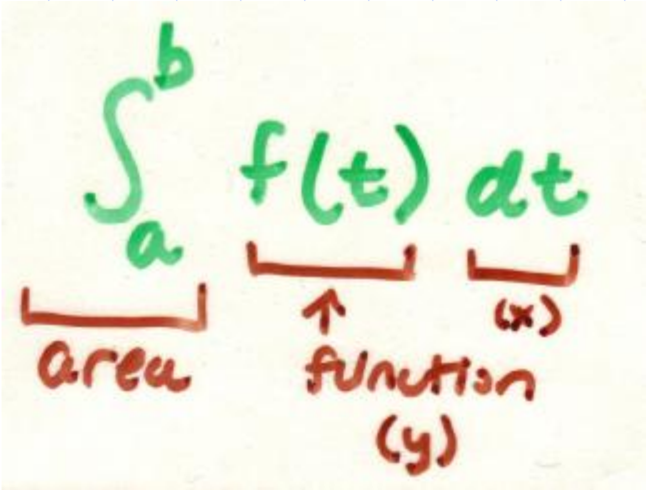


- Note: Area can be + or –
- Ex: Area = negative displacement

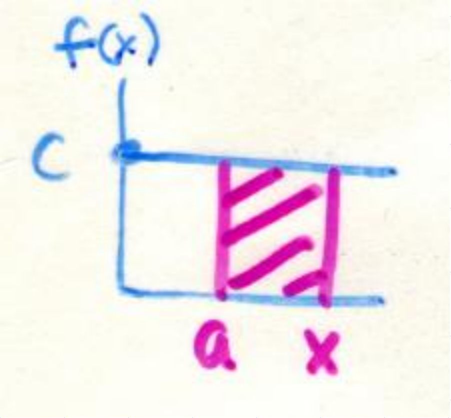


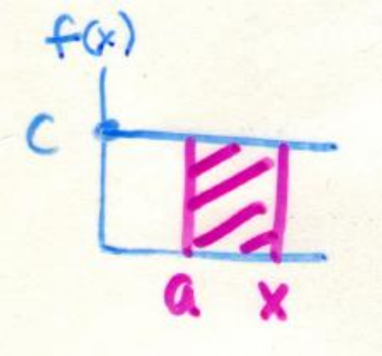
- Find the area under the curve of this!





- Let’s examine this…
- Ex: f(x) = C function is constant





$$A = c(x-a)$$

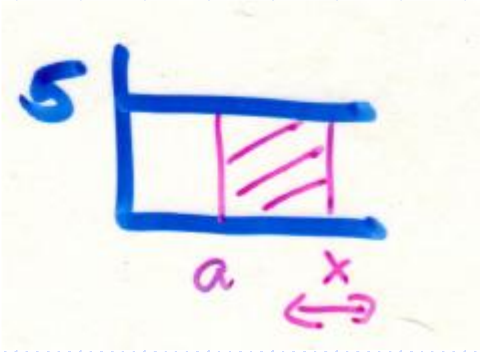
$$A'(x) = \frac{d}{dx}cx - \frac{d}{dx}ca$$

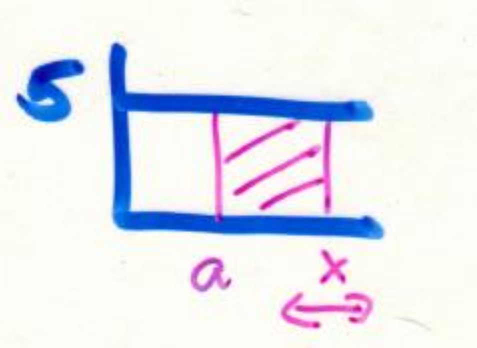
$$\underline{A'(x) = c = f(x)}$$

* The derivative of Area = the original function.

Examples

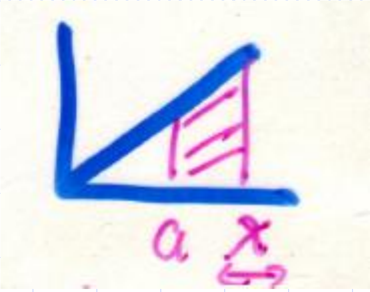
- Examples to illustrate this concept:
- Derivative of Area = original function
- Ex: $f(x) = 5$
  - find area under curve, from “a” to “x”.
  - “a” is constant, “x” is variable and can move.





- Area = (length)(width)
- A = $5x - 5a$
- $A'(x) = 5$
  - derivative of A = original function.

- Ex: $f(x) = 3x$



Area = large Δ – small Δ

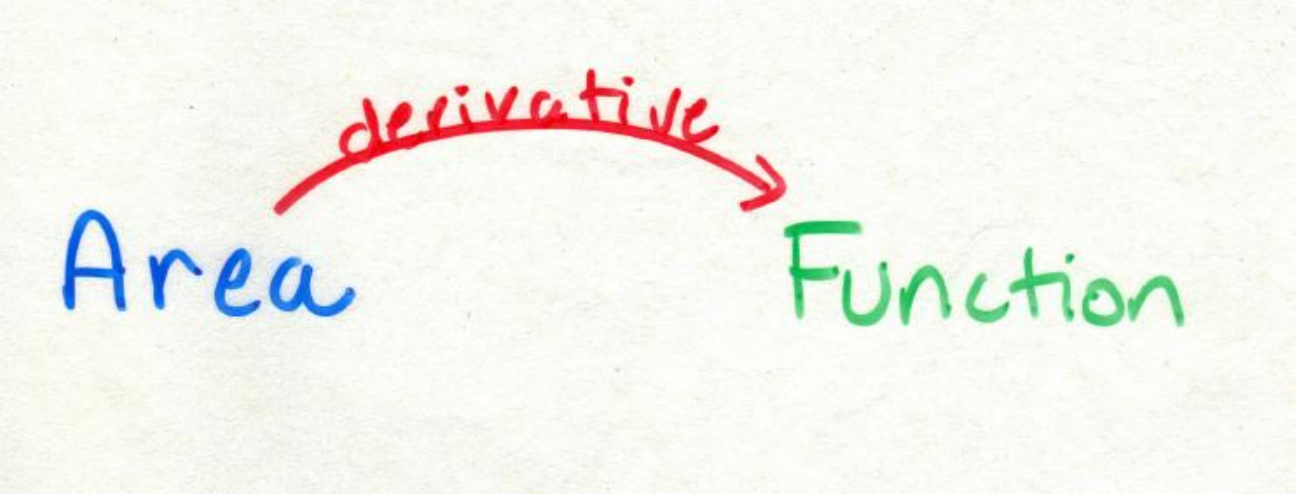
$= \frac{1}{2}(x)(3x) - \frac{1}{2}(a)(3a)$

Area $= (3/2)x^2 - (3/2)a^2$

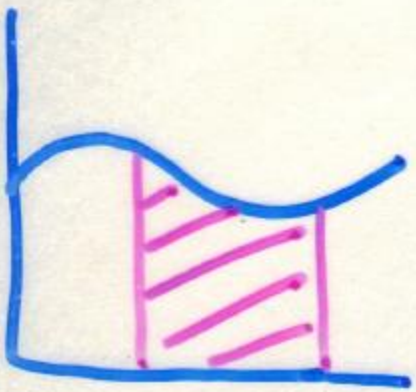
$A'(x) = 3x$

Derivative of Area = original function

- We know the derivative of Area = the original function.
- To find Area, we want the *anti-derivative* (integral) of the function.

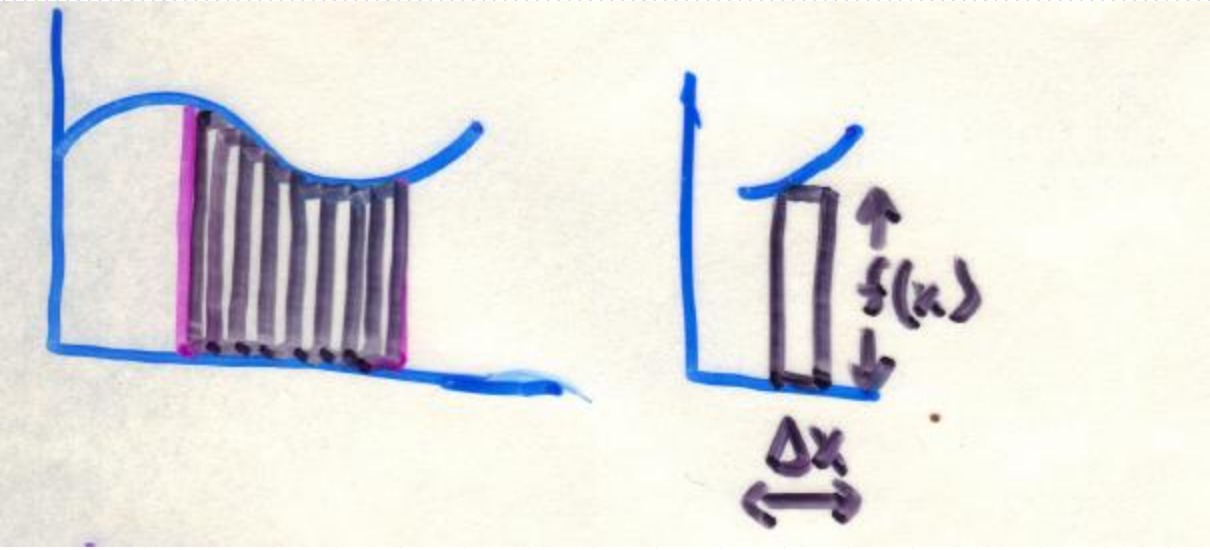


- Given a function, how do we find the "area under the curve"?
- We must "undo" the derivative (or do the inverse…)
- Anti-derivative = Integral
- We Integrate the function!



← How do you get the Area under the curve for a given function?

Answer: Area is approximated as the sum of many small rectangles.



As Δx gets smaller, the approximation is better.
Then, add up the area of all of the small rectangles….

Area under curve $\approx$ Area of rectangle

$\Delta A \approx f(x)\,\Delta x$ (length x width)

As $\Delta x \to 0$, the approximation is better
the little extra triangle Δ shrinks !
(explosion) ---- a Limit !!

$$\frac{dA}{dx} = f(x)$$

differential = a tiny bit of...

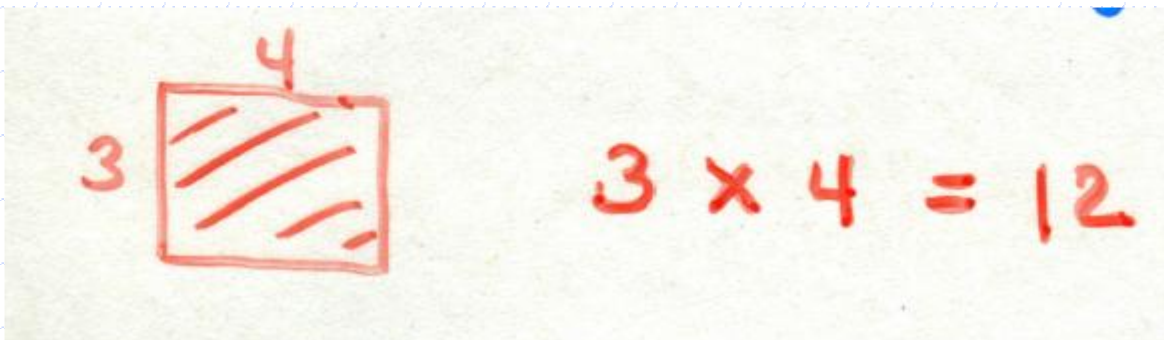
$$dA = f(x)\,dx$$

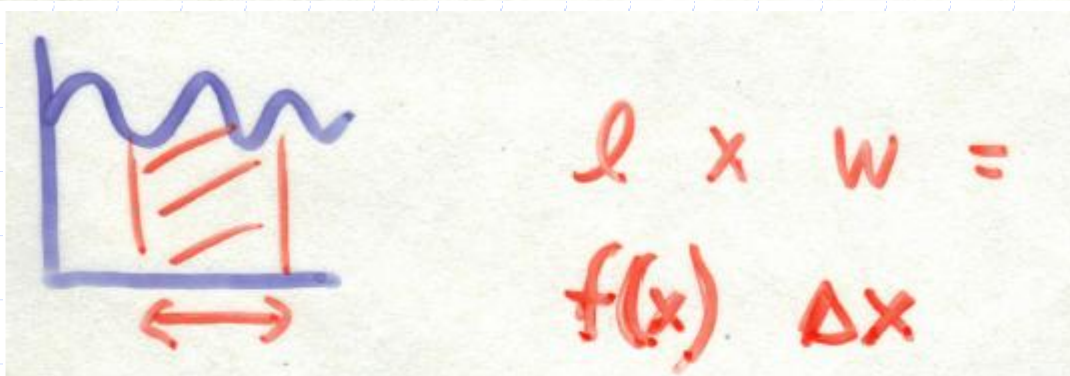
$$A = \int f(x)\,dx$$

Integral (anti-derivative) = sum of all the rectangles.

Another way to think of Integrals –

A way to multiply changing numbers





Integrals

- Integral = a new function whose derivative is the original (given) function.
  - Find the new function whose derivative is the given function.

Ex:

- Find $\int 1$ (the function whose derivative is 1)
  - Answer: x
  - Check by taking the derivative
- BUT– a constant may have dropped out of the derivative, so:

$$\boxed{\int 1 = x + C}$$

Constant of integration.

Ex:

- $\int 6x\,dx$

$$\int 6x\,dx$$

↑ f(x) (height)   ↑ width

Ans: $\boxed{\int 6x\,dx = 3x^2 + C}$ Check - take deriv.

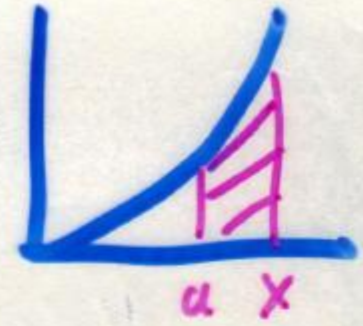
$f(x) = 5 \qquad \int 5 = 5x + C$

$f(x) = 3x \qquad \int 3x = \frac{3}{2}x^2 + C$

"C" = constant of integration.

Check by taking derivative...

Ex 3. $f(x) = x^2$



Area $= \int x^2 = \underline{\qquad\qquad}$

Check by taking derivative.

Ex:

- $\int(x^2+2x+4)dx$

- Answer:

$$\boxed{\frac{1}{3}x^3 + x^2 + 4x + C}$$

Ex:

- $\int(12x^3-4x^2+10x-4)dx$

- Answer:

$$\boxed{3x^4 - \frac{4}{3}x^3 + 5x^2 - 4x + C}$$

Integrals of Trig Functions

$$\frac{d}{dx}\sin x = \cos x$$

$$d\sin x = \cos x\,dx$$

$$\sin x = \int \cos x\,dx$$

$$\boxed{\int \cos x\,dx = \sin x}$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$d\cos x = -\sin x\,dx$$

$$\cos x = -\int \sin x\,dx$$

$$\boxed{\int \sin x\,dx = -\cos x}$$

On AP list:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} \qquad (n \neq -1)$$

assume $+C$

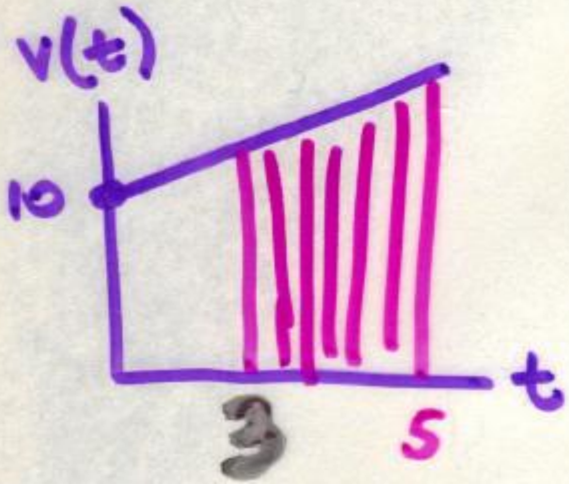
$$\int e^x\,dx = e^x$$

$$\int \frac{dx}{x} = \ln|x|$$

$\int \frac{dx}{x}$ is really $\int \frac{1}{x}\,dx$ ....

inverse function

Physics (Yay!)



$$v(t) = 4t + 10$$

$$\text{slope} = v'(t) = \frac{\Delta v}{\Delta t} = a = \boxed{4\ m/s^2}$$

$$\text{area} = \int_3^5 (4t + 10)\,dt$$

$$= 2t^2 + 10t + C$$

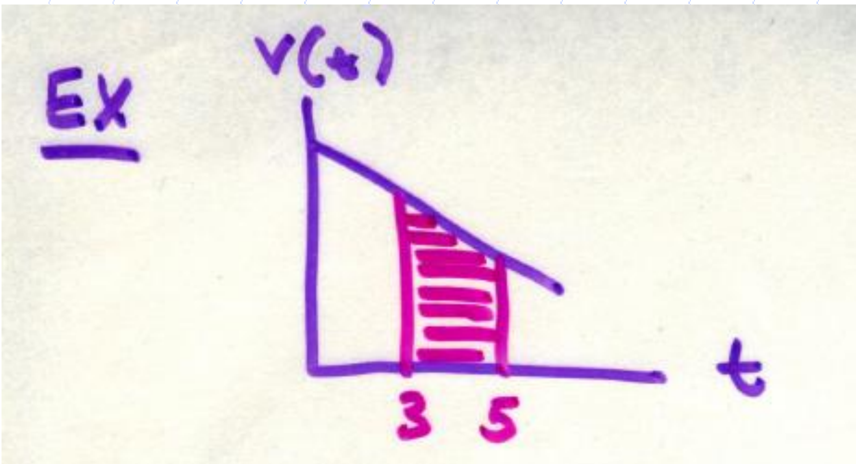
$$= * \left(2(5)^2 + 10(5)\right) - \left(2(3)^2 + 10(3)\right)$$

$$= 100 - 48$$

$$= \boxed{52\ m} = \text{displacement}$$

* Definite Integrals -
Evaluate at (final value) - (initial value).
= $\boxed{\text{answer}}$
vs -
Indefinite Integrals

Ex:



$v(t) = -2t + 20$

accel = ?

displacement (Δx) = ?

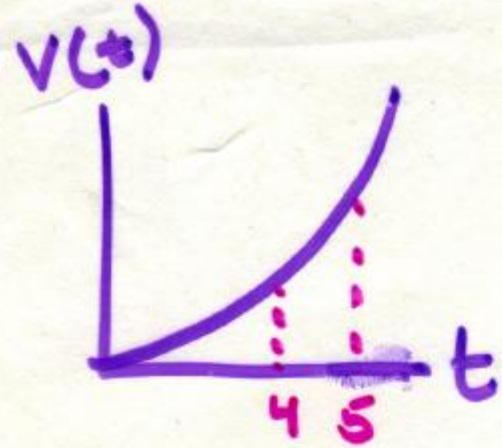
$a = \text{Slope} = v'(t) = \boxed{-2\ m/s^2}$

$$\Delta x = d = \text{Area} = \int_3^5 (-2t + 20)\, dt$$

$$= -t^2 + 20t$$

$$= \boxed{}\ m$$

Ex:



$v(t) = 3t^2$

$a = ?$

d or $\Delta x = ?$ (from 4-5 sec).

On AP list:

$$\int \cos x \, dx = \sin x$$

$$\int \sin x \, dx = -\cos x$$